

INDEPENDENT PARAMETERS FOR SPECIAL INSTANTON BUNDLES ON \mathbb{P}^{2n+1}

NORBERT HOFFMANN

ABSTRACT. Motivated by Yang-Mills theory in $4n$ dimensions, and generalizing the notion due to Atiyah, Drinfeld, Hitchin and Manin for $n = 1$, Okonek, Spindler and Trautmann introduced instanton bundles and special instanton bundles as certain algebraic vector bundles of rank $2n$ on the complex projective space \mathbb{P}^{2n+1} . The moduli space of special instanton bundles is shown to be rational.

1. INTRODUCTION

An instanton is a self-dual solution of the $SU(2)$ Yang-Mills equations on the Euclidean sphere S^4 [9, Exp. 1]. Via the Penrose transformation, these correspond to certain algebraic vector bundles of rank 2 on the complex projective space \mathbb{P}^3 , which are consequently called instanton bundles [2, 9].

Salamon [19] and Corrigan-Goddard-Kent [7] have independently generalized this picture to Yang-Mills theory in dimension $4n$, replacing S^4 and \mathbb{P}^3 by the quaternionic projective space $\mathbb{H}\mathbb{P}^n$ and the twistor space over it, which is the complex projective space \mathbb{P}^{2n+1} . Now certain Yang-Mills connections on $\mathbb{H}\mathbb{P}^n$ correspond to certain algebraic vector bundles on \mathbb{P}^{2n+1} ; cf. [5, §3] and [14, Corollary to Main Theorem 2]. Motivated by this generalization of the Penrose transform, Okonek and Spindler [18] extended the notion of instanton bundles to algebraic vector bundles of rank $2n$ on \mathbb{P}^{2n+1} .

A fundamental tool in the study of algebraic vector bundles on projective spaces is their description in terms of monads [13, 3, 17]. It often allows to describe the vector bundles at hand in terms of some linear algebra data. In particular, the Beilinson spectral sequence [4] yields a correspondence between instanton bundles and instanton monads; we recall this in Section 2 below.

Hirschowitz and Narasimhan [10] have introduced certain special instanton bundles on \mathbb{P}^3 , calling them special ‘t Hooft bundles. They

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have studied their moduli space, and proved in particular that it is rational. Recently Tikhomirov [23] has shown that the moduli space of all instanton bundles on \mathbb{P}^3 with fixed odd c_2 is irreducible, and together with Markushevich also that it is rational [16].

Generalizing from \mathbb{P}^3 to \mathbb{P}^{2n+1} , Spindler and Trautmann [22] have introduced the notion of special instanton bundles on \mathbb{P}^{2n+1} and constructed their moduli space. They also prove that this moduli space is non-empty, irreducible, and smooth of dimension $2n(k+1) + (2n+2)^2 - 7$ if one fixes the quantum number $k+1$ of these instantons. The main result of the present paper, Theorem 4.9, states that the Spindler-Trautmann moduli space of special instanton bundles on \mathbb{P}^{2n+1} is rational. So these instanton bundles depend on $2n(k+1) + (2n+2)^2 - 7$ independent complex parameters.

Like in the case of vector bundles on a curve [15, 12, 11], the proof involves the rationality of some Severi-Brauer varieties, whose Brauer classes are related to the existence of Poincaré families, or universal families, of vector bundles parameterized by the moduli space. Spindler and Trautmann determined in [22] when such Poincaré families of special instanton bundles exist. Somewhat surprisingly, Theorem 4.9 proves rationality of the moduli space even in the cases where there is no Poincaré family. A similar phenomenon has been observed for moduli spaces of vector bundles on rational surfaces [21, 8].

Another main ingredient in the proof is the no-name lemma about rationality of quotients of vector spaces modulo linear groups; cf. for example [6]. It allows us to finally reduce to a rationality problem for quotients modulo PGL_2 , where the invariant ring is known explicitly; that's how we prove rationality of a moduli space without at the same time constructing a Poincaré family on some open part of it. Compared to the special case $n=1$ of bundles on \mathbb{P}^3 , where the rationality is proved in [10], this ingredient is new. Then some results about Severi-Brauer varieties from [10] complete the proof.

The structure of the present text is as follows. In Section 2, we recall the definition of mathematical instanton bundles that we will work with, and we review the correspondence to instanton monads. In Section 3, we recall the notion of special instanton bundles due to Spindler and Trautmann, and we also review their construction of moduli spaces for these; both aspects are based on the correspondence to instanton monads. Section 4 is devoted to the proof of the main result, Theorem 4.9.

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2. MATHEMATICAL INSTANTON BUNDLES

In this section, we recall the notion of (mathematical) instanton bundles, and their description in terms of monads.

We work over the odd-dimensional complex projective space

$$\mathbb{P} := \mathbb{P}^{2n+1}/\mathbb{C}, \quad n \geq 1.$$

An algebraic vector bundle E over \mathbb{P} is said to have *natural cohomology* if there is at most one q with $H^q(\mathbb{P}, E) \neq 0$.

Definition 2.1. Let $k \geq 1$ be an integer. A *k-instanton bundle* is an algebraic vector bundle E over $\mathbb{P} = \mathbb{P}^{2n+1}$ with the following properties:

- i) The rank of E is $2n$.
- ii) The Chern polynomial of E is $c_t(E) = (1 - t^2)^{-k}$.
- iii) For each $l \in \mathbb{Z}$ with $-2n - 1 \leq l \leq 0$, the twisted vector bundle

$$E(l) := E \otimes \mathcal{O}_{\mathbb{P}}(1)^{\otimes l}$$

has natural cohomology.

Remark 2.2. i) Every k -instanton bundle is simple by [1, Theorem 2.8].

ii) Some authors moreover require that E has trivial splitting type. Note that this is an open condition. It is not used here, so we don't include it in our definition.

Let $\mathbf{Cpx}(\mathbb{P})$ denote the category of complexes of coherent $\mathcal{O}_{\mathbb{P}}$ -modules. By a *monad*, we mean a complex of vector bundles

$$E^\bullet = [0 \longrightarrow E^{-1} \xrightarrow{i} E^0 \xrightarrow{p} E^1 \longrightarrow 0] \in \mathbf{Cpx}(\mathbb{P})$$

such that p is surjective, and i is an isomorphism onto a subbundle. As *morphisms* of monads, we take the morphisms in $\mathbf{Cpx}(\mathbb{P})$. The *cohomology* of the monad E^\bullet is the vector bundle

$$E := \ker(p)/\text{image}(i).$$

Definition 2.3. A monad E^\bullet is a *k-instanton monad* for $k \geq 1$, if

$$E^{-1} \cong \mathcal{O}_{\mathbb{P}}(-1)^k, \quad E^0 \cong \mathcal{O}_{\mathbb{P}}^{2n+2k}, \quad \text{and} \quad E^1 \cong \mathcal{O}_{\mathbb{P}}(1)^k.$$

The following standard facts show that the categories of k -instanton bundles and of k -instanton monads are equivalent.

Proposition 2.4. i) If E^\bullet is a *k-instanton monad*, then its cohomology $E = \ker(p)/\text{image}(i)$ is a *k-instanton bundle*.

- ii) If E^\bullet and F^\bullet are two k -instanton monads with cohomologies E and F , then $\mathrm{Hom}_{\mathbb{P}}(E, F) = \mathrm{Hom}_{\mathrm{Cpx}(\mathbb{P})}(E^\bullet, F^\bullet)$.
- iii) Every k -instanton bundle E is isomorphic to the cohomology of some k -instanton monad E^\bullet .

Proof. i) Let E be the cohomology of a k -instanton monad

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}}(-1)^k \xrightarrow{i} \mathcal{O}_{\mathbb{P}}^{2n+2k} \xrightarrow{p} \mathcal{O}_{\mathbb{P}}(1)^k \longrightarrow 0.$$

Then the short exact sequences of vector bundles

$$\begin{aligned} 0 \longrightarrow \ker(p) \longrightarrow \mathcal{O}_{\mathbb{P}}^{2n+2k} \xrightarrow{p} \mathcal{O}_{\mathbb{P}}(1)^k \longrightarrow 0, \\ 0 \longrightarrow \mathcal{O}_{\mathbb{P}}(-1)^k \xrightarrow{i} \ker(p) \longrightarrow E \longrightarrow 0 \end{aligned}$$

show that E has rank $2n$ and Chern polynomial $c_t(E) = (1 - t^2)^{-k}$. Now tensor these sequences with $\mathcal{O}_{\mathbb{P}}(l)$, and consider the resulting long exact cohomology sequences. For $l = 0$, we get an exact sequence

$$H^0(\ker(p)) \longrightarrow H^0(E) \longrightarrow H^1(\mathcal{O}_{\mathbb{P}}(-1)^k) = 0.$$

Here $H^0(\ker(p)) = 0$, since $\ker(p)$ is stable by [1, Theorem 2.8]; hence $H^0(E) = 0$. For general l , we see that $H^q(E(l))$ vanishes whenever

$$H^{q-1}(\mathcal{O}_{\mathbb{P}}(l+1)) = H^q(\mathcal{O}_{\mathbb{P}}(l)) = H^{q+1}(\mathcal{O}_{\mathbb{P}}(l-1)) = 0.$$

The latter holds in each of the following four cases:

- $l = -2n - 1$, and $q \neq 2n$
- $-2n \leq l \leq -2$, and q arbitrary
- $l = -1$, and $q \neq 1$
- $l = 0$, and $q \geq 2$

It follows that $E(l)$ has natural cohomology for $-2n - 1 \leq l \leq 0$, the only possibly nonzero cohomology groups in this range being

$$H^{2n}(E(-2n-1)), \quad H^1(E(-1)), \quad \text{and} \quad H^1(E).$$

ii) Let E^\bullet and F^\bullet more generally be two monads, with cohomologies E and F . Given a morphism $E \rightarrow F$, we try to lift it to a morphism of monads $E^\bullet \rightarrow F^\bullet$. As explained in [17, Lemma II.4.1.3], the obstructions against the existence and uniqueness of such a lift are some classes in

$$\mathrm{Ext}^q(E^i, F^j) = H^q(\mathbb{P}, (E^i)^{\mathrm{dual}} \otimes F^j)$$

with $i > j$ and $q \leq 2$. If E^\bullet and F^\bullet are k -instanton monads, then

$$\mathrm{Ext}^q(E^i, F^j) = H^q(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(j-i)^{\mathrm{rank}(E_i) \cdot \mathrm{rank}(E_j)})$$

vanishes for all $i > j$ and all $q \leq 2$.

iii) For any vector bundle E over \mathbb{P} , Beilinson [4] has constructed a spectral sequence with E_1 -term

$$E_1^{pq} = H^q(\mathbb{P}, E(p)) \otimes \Omega_{\mathbb{P}}^{-p}(-p)$$

which converges to E . If E is a k -instanton bundle, then most of these terms vanish, and the claim follows; for some more details, see for example [18, Lemma 1.3 and Corollary 1.4]¹. \square

Remark 2.5. Let S be a scheme of finite type over \mathbb{C} .

A vector bundle \mathcal{E} over $\mathbb{P} \times S$ is a *family of k -instanton bundles* if its restriction to every closed point $s \in S(\mathbb{C})$ is a k -instanton bundle.

A monad \mathcal{E}^\bullet of vector bundles over $\mathbb{P} \times S$ is a *family of k -instanton monads* if its restriction to every closed point $s \in S(\mathbb{C})$ is a k -instanton monad. (This implies $\mathcal{E}^d \cong \mathcal{O}_{\mathbb{P}}(d) \boxtimes F^d$ for vector bundles F^d over S .)

With these definitions, the above equivalence between instanton bundles and instanton monads extends to families; cf. [22, p. 585f.].

3. MODULI OF SPECIAL INSTANTONS

This section recalls the notion of *special* instantons due to Spindler-Trautmann [22], and their construction of moduli spaces for these.

Still assuming $k \geq 1$, we fix two complex vector spaces $V, W \cong \mathbb{C}^2$. Let $V^{\otimes r} \twoheadrightarrow S^r V$ denote the r th symmetric power of V . Then

$$\begin{aligned} E_{n,k}^0 &:= (S^{n+k} V \otimes W)^{\text{dual}} \otimes \mathcal{O}_{\mathbb{P}} \quad \text{and} \\ E_{n,k}^1 &:= (S^k V)^{\text{dual}} \otimes \mathcal{O}_{\mathbb{P}}(1) \end{aligned}$$

are vector bundles over $\mathbb{P} = \mathbb{P}^{2n+1}$, with ranks $2n + 2(k+1)$ and $k+1$.

The multiplication $\mu_r : S^r V \otimes S^n V \rightarrow S^{n+r} V$ induces a linear map

$$\mu_r^* : (S^{n+r} V)^{\text{dual}} \longrightarrow (S^r V)^{\text{dual}} \otimes (S^n V)^{\text{dual}}.$$

The choice of a linear isomorphism

$$b : (S^n V \otimes W)^{\text{dual}} \xrightarrow{\sim} H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))$$

thus determines a morphism of vector bundles

$$p_b : E_{n,k}^0 \longrightarrow E_{n,k}^1$$

such that $H^0(p_b)$ is the composition

$$H^0(E_{n,k}^0) \xrightarrow{\mu_k^*} (S^k V)^{\text{dual}} \otimes (S^n V \otimes W)^{\text{dual}} \xrightarrow{b} H^0(E_{n,k}^1).$$

Note that this composition is injective, since μ_k is surjective.

¹In [18], all instanton bundles are assumed to be symplectic. However, this assumption is not used in the quoted Lemma 1.3 and Corollary 1.4.

Definition 3.1. i) A $(k+1)$ -instanton monad $E^\bullet \in \mathbf{Cpx}(\mathbb{P})$ is called *special* if its stupid truncation

$$\tau^{\geq 0}(E^\bullet) := [E^0 \xrightarrow{p} E^1]$$

is in $\mathbf{Cpx}(\mathbb{P})$ isomorphic to a complex of the form

$$[E_{n,k}^0 \xrightarrow{p_b} E_{n,k}^1]$$

for some isomorphism $b : (S^n V \otimes W)^{\text{dual}} \rightarrow H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))$.

ii) A $(k+1)$ -instanton bundle E over \mathbb{P} is *special* if E is isomorphic to the cohomology of some special $(k+1)$ -instanton monad.

Remark 3.2. This definition is equivalent to the original definition given by Spindler and Trautmann, according to [22, Proposition 4.2].

In particular, all these special instanton bundles are simple, and have trivial splitting type; cf. also [22, Proposition 4.5].

With $k \geq 1$ still fixed, we consider all complexes of the form

$$[E_{n,k}^0 \xrightarrow{p_b} E_{n,k}^1] \in \mathbf{Cpx}(\mathbb{P})$$

as above. The next step is to classify them up to isomorphism in $\mathbf{Cpx}(\mathbb{P})$.

We have an exact sequence of algebraic groups

$$1 \longrightarrow \mathbb{G}_m \xrightarrow{\iota_n} \text{GL}(V) \times \text{GL}(W) \xrightarrow{\pi_n} \text{GL}(S^n V \otimes W),$$

defined by $\iota_n(\lambda) := (\lambda \text{id}_V, \lambda^{-n} \text{id}_W)$ and $\pi_n(\alpha, \beta) := S^n \alpha \otimes \beta$. In particular, π_n allows us to identify the 7-dimensional group

$$G_n := \text{GL}(V) \times \text{GL}(W) / \iota_n(\mathbb{G}_m)$$

with a closed subgroup of $\text{GL}(S^n V \otimes W)$.

Proposition 3.3. *Let $k \geq 1$ be given. The homogeneous variety*

$$X_n := G_n \backslash \text{GL}(S^n V \otimes W)$$

is a coarse moduli space for complexes of the form $[E_{n,k}^0 \xrightarrow{p_b} E_{n,k}^1]$.

Proof. Given an isomorphism $b : (S^n V \otimes W)^{\text{dual}} \rightarrow H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))$ and an element $g \in \text{GL}(S^n V \otimes W)$, we get another isomorphism

$$g \cdot b := b \circ g^{\text{dual}} : (S^n V \otimes W)^{\text{dual}} \xrightarrow{\sim} H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)).$$

This defines a simply transitive action of $\text{GL}(S^n V \otimes W)$ on the set of all such isomorphisms b .

Every pair $(\alpha, \beta) \in \mathrm{GL}(V) \times \mathrm{GL}(W)$ yields a commutative diagram

$$\begin{array}{ccc} (S^{n+k}V \otimes W)^{\mathrm{dual}} & \xrightarrow{b \circ \mu_k^*} & (S^kV)^{\mathrm{dual}} \otimes H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)) \\ \downarrow (S^{n+k}\alpha^{-1} \otimes \beta^{-1})^{\mathrm{dual}} & & \downarrow (S^k\alpha^{-1})^{\mathrm{dual}} \\ (S^{n+k}V \otimes W)^{\mathrm{dual}} & \xrightarrow{b' \circ \mu_k^*} & (S^kV)^{\mathrm{dual}} \otimes H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)) \end{array}$$

with $b' := \pi_n(\alpha, \beta) \cdot b$, and hence an isomorphism of complexes

$$\begin{array}{ccc} E_{n,k}^0 & \xrightarrow{p_b} & E_{n,k}^1 \\ \downarrow & & \downarrow \\ E_{n,k}^0 & \xrightarrow{p_{b'}} & E_{n,k}^1. \end{array}$$

Conversely, suppose that $[E_{n,k}^0 \xrightarrow{p_b} E_{n,k}^1]$ and $[E_{n,k}^0 \xrightarrow{p_{b'}} E_{n,k}^1]$ are isomorphic in $\mathbf{Cpx}(\mathbb{P})$ for some isomorphisms b and b' . Then there is a pair $(\alpha, \beta) \in \mathrm{GL}(V) \times \mathrm{GL}(W)$ with $b' = \pi_n(\alpha, \beta) \cdot b$, according to step 2) in the proof of [22, Proposition 6.1].

Pick one isomorphism $b_0 : (S^nV \otimes W)^{\mathrm{dual}} \rightarrow H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))$. Assign to each complex $p_{g \cdot b_0} : [E_{n,k}^0 \rightarrow E_{n,k}^1]$ with $g \in \mathrm{GL}(S^nV \otimes W)$ the moduli point $G_n g$ in the coset space $X_n = G_n \backslash \mathrm{GL}(S^nV \otimes W)$. The coset $G_n g$ depends only on the isomorphism class of the complex. This turns X_n into a coarse moduli space for the complexes in question. \square

The group G_n acts via its quotient $\mathrm{PGL}(V)$ on $\mathbb{P}V := (V \setminus \{0\})/\mathbb{G}_m$. Thus the homogeneous variety X_n carries a natural conic bundle

$$C := \mathbb{P}V \times^{G_n} \mathrm{GL}(S^nV \times W) \longrightarrow X_n$$

with fibers $\mathbb{P}V$. The fiberwise symmetric power

$$C^{(2n+k)} \longrightarrow X_n$$

is a projective bundle with fibers

$$(\mathbb{P}V)^{(2n+k)} = \mathbb{P}H^0(\mathbb{P}V, \mathcal{O}(2n+k)) = \mathbb{P}(S^{2n+k}V)^{\mathrm{dual}}.$$

Note that every element $f \in (S^{2n+k}V)^{\mathrm{dual}}$ induces a linear map

$$\mu_{n+k}^*(f) : S^nV \longrightarrow (S^{n+k}V)^{\mathrm{dual}}.$$

We form the associated Grassmannian bundle

$$\mathrm{Gr}_k(C^{(2n+k)}) \longrightarrow X_n,$$

which parameterizes linear subspaces $\mathbb{P}U \subseteq (\mathbb{P}V)^{(2n+k)}$ of dimension k , or equivalently linear subspaces $U \subseteq (S^{2n+k}V)^{\mathrm{dual}}$ of dimension $k+1$.

Theorem 3.4 (Spindler-Trautmann). *Given $k \geq 1$, let*

$$M_{\mathbb{P}}(k+1) \subseteq \mathrm{Gr}_k(C^{(2n+k)}) \longrightarrow X_n$$

denote the open locus of all linear subspaces $U \subseteq (S^{2n+k}V)^{\mathrm{dual}}$ such that $\mu_{n+k}^(f)$ is injective for all $0 \neq f \in U$. Then $M_{\mathbb{P}}(k+1)$ is a coarse moduli space for special $(k+1)$ -instanton bundles over $\mathbb{P} = \mathbb{P}^{2n+1}$.*

Proof. This statement is Theorem 6.3 in [22]. For the convenience of the reader, we give an outline of the proof.

The starting point is that every special $(k+1)$ -instanton bundle defines a special $(k+1)$ -instanton monad, and hence by truncation a point in X_n . The remaining part of the monad will then be parameterized by the fiber of $M_{\mathbb{P}}(k+1)$ over this point in X_n .

To be more specific, note that the choice of an isomorphism

$$b : (S^n V \otimes W)^{\mathrm{dual}} \xrightarrow{\sim} H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))$$

yields a commutative diagram

$$\begin{array}{ccc} (S^{n+k}V \otimes W \otimes S^n V \otimes W)^{\mathrm{dual}} & \xrightarrow{\mu_k^*} & (S^k V)^{\mathrm{dual}} \otimes S^2((S^n V \otimes W)^{\mathrm{dual}}) \\ b \downarrow & & \downarrow S^2 b \\ \mathrm{Hom}_{\mathbb{P}}(\mathcal{O}_{\mathbb{P}}(-1), E_{n,k}^0) & \xrightarrow{(p_b)_*} & \mathrm{Hom}_{\mathbb{P}}(\mathcal{O}_{\mathbb{P}}(-1), E_{n,k}^1). \end{array}$$

The kernel of the horizontal map μ_k^* consists of all multilinear forms

$$\varphi : S^{n+k}V \otimes W \otimes S^n V \otimes W \longrightarrow \mathbb{C}$$

which satisfy the condition

$$\begin{aligned} & -\varphi(v_1, \dots, v_n, v_{n+1}, \dots, v_{n+k}, w, v'_1, \dots, v'_n, w') \\ & = \varphi(v'_1, \dots, v'_n, v_{n+1}, \dots, v_{n+k}, w', v_1, \dots, v_n, w) \end{aligned}$$

for all $v_i, v'_j \in V$ and $w, w' \in W$. Using that φ is also symmetric in v_1, \dots, v_{n+k} and in v'_1, \dots, v'_n separately, it is easy to deduce that φ is symmetric in all the v 's and alternating in the w 's. This proves

$$\ker(\mu_k^*) = (S^{2n+k}V \otimes \Lambda^2 W)^{\mathrm{dual}}.$$

Hence b induces, via the above diagram, an isomorphism

$$(S^{2n+k}V \otimes \Lambda^2 W)^{\mathrm{dual}} \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{P}}(\mathcal{O}_{\mathbb{P}}(-1), \ker(p_b));$$

cf. also [22, Proposition 3.2]. Given a special $(k+1)$ -instanton monad

$$E^\bullet = [0 \longrightarrow \mathcal{O}_{\mathbb{P}}(-1)^{(k+1)} \xrightarrow{i} E_{n,k}^0 \xrightarrow{p_b} E_{n,k}^1 \longrightarrow 0] \in \mathrm{Cpx}(\mathbb{P}),$$

the components $\mathcal{O}_{\mathbb{P}}(-1) \rightarrow \ker(p_b)$ of i thus span a linear subspace of $(S^{2n+k}V \oplus \Lambda^2 W)^{\text{dual}}$, which corresponds to a linear subspace

$$U \subseteq (S^{2n+k}V)^{\text{dual}}.$$

Since i is an isomorphism onto a subbundle, U has dimension $k+1$, and $\mu_{n+k}^*(f)$ is injective for all $0 \neq f \in U$; cf. [22, (3.7)]. In this way, the special $(k+1)$ -instanton monad E^\bullet defines a point in $M_{\mathbb{P}}(k+1)$.

Conversely, every linear subspace $U \subseteq (S^{2n+k}V)^{\text{dual}}$ of dimension $k+1$ yields, by means of the above isomorphism, a linear subspace

$$(\Lambda^2 W)^{\text{dual}} \otimes U \subseteq \text{Hom}_{\mathbb{P}}(\mathcal{O}(-1), \ker(p_b))$$

of dimension $k+1$, and hence a morphism of vector bundles

$$i_U : \mathcal{O}_{\mathbb{P}}(-1)^{(k+1)} \cong (\Lambda^2 W)^{\text{dual}} \otimes U \otimes \mathcal{O}_{\mathbb{P}}(-1) \longrightarrow \ker(p_b).$$

If $\mu_{n+k}^*(f)$ is injective for all $0 \neq f \in U$, then i_U is an isomorphism onto a subbundle of $\ker(p_b)$, so

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}}(-1)^{(k+1)} \xrightarrow{i_U} E_{n,k}^0 \xrightarrow{p_b} E_{n,k}^1 \longrightarrow 0$$

is a special $(k+1)$ -instanton monad. This shows that $M_{\mathbb{P}}(k+1)$ is a coarse moduli scheme for special $(k+1)$ -instanton monads, and hence also for special $(k+1)$ -instanton bundles due to Proposition 2.4. \square

Remark 3.5. To make this precise using the formalism of moduli functors, one would have to define what a *family* of special $(k+1)$ -instanton bundles is, say parameterized by a scheme S of finite type over \mathbb{C} .

A family of isomorphisms $b : (S^n V \otimes W)^{\text{dual}} \rightarrow H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))$ is an isomorphism of the corresponding trivial vector bundles over S ; it induces as before a complex of vector bundles

$$[E_{n,k}^0 \boxtimes \mathcal{O}_S \longrightarrow E_{n,k}^1 \boxtimes \mathcal{O}_S]$$

over $\mathbb{P} \times S$. One could define that a family \mathcal{E}^\bullet of $(k+1)$ -instanton monads is special if its truncation $\tau^{\geq 0}(\mathcal{E}^\bullet)$ is étale-locally in S isomorphic to a complex of this form, and that a family of $(k+1)$ -instanton bundles \mathcal{E} is special if the corresponding family of $(k+1)$ -instanton monads is. Then the arguments in the above proof of Theorem 3.4 extend routinely to families of special instantons.

Note however that the moduli functor in [22, p. 585] is slightly different from this one for non-reduced S , since they only require that the restriction of \mathcal{E} to all closed points $s \in S(\mathbb{C})$ is special.

Remark 3.6. The moduli space $M_{\mathbb{P}}(k+1)$ is non-empty by [22, 3.7]. It is by construction an irreducible smooth variety of dimension

$$\dim M_{\mathbb{P}}(k+1) = 2n(k+1) + \dim X_n = 2n(k+1) + (2n+2)^2 - 7.$$

4. RATIONALITY

Let G be a linear algebraic group over \mathbb{C} . Suppose that G acts on an integral algebraic variety X of finite type over \mathbb{C} . We denote by $\mathbb{C}(X)^G$ the field of G -invariant rational functions on X .

Lemma 4.1. *Suppose that G acts on the integral variety X' over \mathbb{C} as well, and that there is an open orbit $Gx' \subseteq X'$ with $x' \in X'(\mathbb{C})$. Then*

$$\mathbb{C}(X \times X')^G \cong \mathbb{C}(X)^{\text{Stab}_G(x')}.$$

Proof. This is a special case of the standard ‘lemma of Seshadri’, or ‘slice lemma’; cf. for example [6, Theorem 3.1]. \square

The action of G on X is called *almost free* if there is a dense open subvariety $X^0 \subseteq X$ such that the stabiliser subgroup $\text{Stab}_G(x) \subseteq G$ is trivial for each closed point $x \in X^0(\mathbb{C})$.

Lemma 4.2. *Suppose $V \cong \mathbb{C}^2$, and $n \geq 2$. The natural action of $\text{PGL}(V)$ on the Grassmannian $\text{Gr}_2(S^n V \oplus S^n V)$ is almost free.*

Proof. Suppose that $\alpha \in \text{GL}(V)$ represents a nontrivial element in $\text{PGL}(V)$. Up to multiplication by \mathbb{C}^* , and the choice of an appropriate basis for V , there are three cases:

- 1.) $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}, \quad \lambda \in \mathbb{C}^*, \quad \lambda^r \neq 1 \text{ for all } r \in \{1, \dots, n\},$
- 2.) $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix}, \quad \zeta \text{ a primitive } r\text{th root of unity, } \quad 2 \leq r \leq n,$
- 3.) $\alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$

In each case, let us estimate the dimension of the fixed point set

$$\text{Gr}_2(S^n V \oplus S^n V)^\alpha.$$

In case 1.), $S^n V$ decomposes into 1-dimensional eigenspaces under the semisimple endomorphism α , so $S^n V \oplus S^n V$ decomposes into 2-dimensional eigenspaces. Every 2-dimensional α -invariant subspace

$$U \subseteq S^n V \oplus S^n V$$

is either one of these eigenspaces, or a direct sum of lines in two of them. The former are parameterized by a finite set, the latter by a finite union of products $\mathbb{P}^1 \times \mathbb{P}^1$. Hence we conclude in this case

$$\dim \text{Gr}_2(S^n V \oplus S^n V)^\alpha = 2.$$

The union of these fixed loci has dimension $\leq 2 + \dim \text{PGL}(V) = 5$.

In case 2.), $S^n V$ decomposes into r eigenspaces $(S^n V)_\chi$ under the semisimple endomorphism α . Their dimension is

$$\dim(S^n V)_\chi \leq \lceil (n+1)/r \rceil \leq \lceil (n+1)/2 \rceil \leq (n+2)/2.$$

So the eigenspace $(S^n V)_\chi^2$ of $S^n V \oplus S^n V$ has dimension $\leq n+2$, and the space of 2-dimensional subspaces $U \subseteq (S^n V)_\chi^2$ has dimension

$$\dim \operatorname{Gr}_2((S^n V)_\chi^2) \leq 2n.$$

If $\chi_1 \neq \chi_2$ are two eigenvalues of α on $S^n V$, then

$$\dim(S^n V)_{\chi_1} + \dim(S^n V)_{\chi_2} \leq \dim S^n V = n+1.$$

So the space of direct sums $U = U_1 \oplus U_2$, where U_i is a line in the eigenspace $(S^n V)_{\chi_i}^2$ of α on $S^n V \oplus S^n V$, has dimension

$$\dim \mathbb{P}((S^n V)_{\chi_1}^2) \times \mathbb{P}((S^n V)_{\chi_2}^2) \leq 2 \dim(S^n V) - 2 = 2n.$$

These two arguments cover all 2-dimensional α -invariant subspaces $U \subseteq S^n V \oplus S^n V$. Hence we can conclude in this case

$$\dim \operatorname{Gr}_2(S^n V \oplus S^n V)^\alpha \leq 2n.$$

Such $\alpha \in \operatorname{GL}(V)$ yield finitely many 2-dimensional conjugacy classes in $\operatorname{PGL}(V)$. So the union of these fixed loci has dimension $\leq 2n+2$.

In case 3.), we can choose a basis $x, y \in V$ with

$$\alpha(x) = x \quad \text{and} \quad \alpha(y) = x + y.$$

The endomorphism $(\alpha - \operatorname{id})^r$ of $S^n V$ has kernel

$$\ker(\alpha - \operatorname{id})^r = x^{n-r+1} \cdot S^{r-1} V$$

for $1 \leq r \leq n+1$. In particular, the kernel of $(\alpha - \operatorname{id})^r$ on $S^n V \oplus S^n V$ has dimension $2r$. Suppose that $U \subseteq S^n V \oplus S^n V$ is 2-dimensional and α -invariant. It follows that $\alpha - \operatorname{id}$ is a nilpotent endomorphism of U .

If $\alpha - \operatorname{id} = 0$ on U , then U coincides with the kernel $\mathbb{C} \cdot x^n \oplus \mathbb{C} \cdot x^n$ of $\alpha - \operatorname{id}$ on $S^n V \oplus S^n V$; hence U is unique in this situation.

Otherwise, we have $(\alpha - \operatorname{id})^2 = 0$ on U , and $U = \mathbb{C} \cdot u \oplus \mathbb{C} \cdot \alpha(u)$ for any element $u \in U$ with $\alpha(u) \neq u$. Here u can be any element in the 4-dimensional kernel of $(\alpha - \operatorname{id})^2$ on $S^n V \oplus S^n V$ with $\alpha(u) \neq u$.

Hence we conclude in this case

$$\dim \operatorname{Gr}_2(S^n V \oplus S^n V)^\alpha = 4 - 2 = 2.$$

This $\alpha \in \operatorname{GL}(V)$ yields one 2-dimensional conjugacy class in $\operatorname{PGL}(V)$. So the union of these fixed loci has dimension $\leq 2 + 2 = 4$.

The Grassmannian $\operatorname{Gr}_2(S^n V \oplus S^n V)$ has dimension $4n$. Using the assumption $n \geq 2$, we see that the closure of all these fixed points has smaller dimension. On the open complement, $\operatorname{PGL}(V)$ acts freely. \square

We say that the field $\mathbb{C}(X)^G$ of G -invariant rational functions on X is *rational* if it is purely transcendental over the base field \mathbb{C} .

Example 4.3. The group $\mathrm{PGL}(V)$ with $V \cong \mathbb{C}^2$ acts on the vector space $\mathrm{End}(V)^2$ over \mathbb{C} by simultaneous conjugation. The action is known to be almost free, and the field of invariants

$$(1) \quad \mathbb{C}(\mathrm{End}(V)^2)^{\mathrm{PGL}(V)}$$

is rational. In fact, sending $(\alpha_1, \alpha_2) \in \mathrm{End}(V)^2$ to the traces of the five maps $\alpha_1, \alpha_2, \alpha_1^2, \alpha_1\alpha_2, \alpha_2^2 \in \mathrm{End}(V)$ defines an isomorphism

$$\mathrm{End}(V)^2 // \mathrm{PGL}(V) \xrightarrow{\sim} \mathbb{A}^5.$$

Lemma 4.4 (No-name lemma). *Let G act linearly on vector spaces M and M' of finite dimension over \mathbb{C} . If G is reductive, and acts almost freely on M , then $\mathbb{C}(M \oplus M')^G$ is purely transcendental over $\mathbb{C}(M)^G$.*

Proof. This statement is contained for example in [6, Corollary 3.8]. \square

Example 4.5. Let the group $\mathrm{PGL}(V)$ with $V \cong \mathbb{C}^2$ act linearly on a finite-dimensional vector space M' over \mathbb{C} . Then the field of invariants

$$\mathbb{C}(\mathrm{End}(V)^2 \oplus M')^{\mathrm{PGL}(V)}$$

is purely transcendental over the field (1), and hence rational.

We keep the notation of the previous section, so $V, W \cong \mathbb{C}^2$, and

$$G_n = \mathrm{GL}(V) \times \mathrm{GL}(W) / \iota_n(\mathbb{G}_m) \quad \text{with} \quad \iota_n(\lambda) := (\lambda \mathrm{id}_V, \lambda^{-n} \mathrm{id}_W).$$

Proposition 4.6. *The coarse moduli space $X_n = G_n \backslash \mathrm{GL}(S^n V \otimes W)$ constructed in Proposition 3.3 is rational.*

Proof. The function field of X_n is by construction

$$\mathbb{C}(X_n) \cong \mathbb{C}((S^n V \otimes W)^{2n+2})^{G_n}.$$

We start with the special case $n = 1$. The action of G_1 on $V \otimes W$ has an open orbit, whose points correspond to linear maps $\psi : W^{\mathrm{dual}} \rightarrow V$ that are bijective. Thus Lemma 4.1 yields

$$\mathbb{C}(X_1) \cong \mathbb{C}((V \otimes W)^4)^{G_1} \cong \mathbb{C}((V \otimes W)^3)^{\mathrm{Stab}_{G_1}(\psi)}.$$

The canonical projection $G_1 \rightarrow \mathrm{PGL}(V)$ restricts to an isomorphism

$$\mathrm{Stab}_{G_1}(\psi) \xrightarrow{\sim} \mathrm{PGL}(V).$$

Since the linear isomorphism

$$V \otimes W \xrightarrow{(\psi^{-1})^{\mathrm{dual}}} V \otimes V^{\mathrm{dual}} = \mathrm{End}(V)$$

intertwines the action of $\text{Stab}_{G_1}(\psi)$ with that of $\text{PGL}(V)$, we conclude

$$\mathbb{C}(X_1) \cong \mathbb{C}(\text{End}(V)^3)^{\text{PGL}(V)}.$$

Using Example 4.5, it follows that X_1 is rational.

For the rest of the proof, we assume $n \geq 2$. The group $\text{GL}(V)$ acts on the 2-dimensional vector space

$$V(n) := \begin{cases} \mathbb{C}^2 \otimes \det^{\frac{n}{2}} V & \text{for } n \text{ even,} \\ V \otimes \det^{\frac{n-1}{2}} V & \text{for } n \text{ odd,} \end{cases}$$

in such a way that the center $\mathbb{G}_m \subseteq \text{GL}(V)$ acts with weight n . Thus the action of $\text{GL}(V) \times \text{GL}(W)$ on the 4-dimensional vector space

$$V(n) \otimes W$$

descends to an action of G_n . This action again has an open orbit, whose points correspond to bijective linear maps $\psi : W^{\text{dual}} \rightarrow V(n)$.

Viewing elements of $(S^n V \otimes W)^2$ as linear maps from W^{dual} to the direct sum $S^n V \oplus S^n V$, the open locus of injective linear maps is a $\text{GL}(W)$ -torsor over the Grassmannian $\text{Gr}_2(S^n V \oplus S^n V)$. Thus G_n acts almost freely on $(S^n V \otimes W)^2$, due to Lemma 4.2.

It follows that the field $\mathbb{C}(X_n)$ is purely transcendental over the field

$$\mathbb{C}((S^n V \otimes W)^2 \oplus \text{End}(V)^2 \oplus (V(n) \otimes W))^{G_n},$$

since both are purely transcendental over $\mathbb{C}((S^n V \otimes W)^2)^{G_n}$ according to Lemma 4.4, the former of transcendence degree $2n(2n+2) \geq 24$, and the latter of transcendence degree 12.

Thus Lemma 4.1 yields that $\mathbb{C}(X_n)$ is purely transcendental over

$$\mathbb{C}((S^n V \otimes W)^2 \oplus \text{End}(V)^2)^{\text{Stab}_{G_n}(\psi)}.$$

But this field is rational according to Example 4.5, since the stabiliser of ψ in G_n projects again isomorphically onto $\text{PGL}(V)$. \square

Remark 4.7. The conic bundle $C \rightarrow X_n$ with fibers $\mathbb{P}V$ is not Zariski-locally trivial; cf. [22, Proposition 8.5]. It follows that the G_n -torsor $\text{GL}(S^n V \otimes W) \rightarrow X_n$ is not Zariski-locally trivial either. The obstruction against both is a Brauer class, which can be described as follows:

The proof of Proposition 4.6 shows that $\mathbb{C}(X_n)$ is actually purely transcendental over $\mathbb{C}(\text{End}(V)^2)^{\text{PGL}(V)}$. Over the latter, one has the so-called generic quaternion algebra; cf. [20, Chapter 14]. Its image in the Brauer group of $\mathbb{C}(X_n)$ is the obstruction class in question.

An equivalent way to state this is to say that the stack quotient of $\text{GL}(S^n V \otimes W)$ modulo $\text{GL}(V) \times \text{GL}(W)$ is birational to an affine

space times the stack quotient of $\text{End}(V)^2$ modulo $\text{GL}(V)$. This can be proved along the same lines as Proposition 4.6 above.

Remark 4.8. The variety X_n , its rationality, and the local nontriviality of bundles over it in Remark 4.7, did not depend on k . But the existence of Poincaré families over $\mathbb{P} \times X_n$ does depend on k , as follows.

Fix $k \geq 1$. The closed points $x \in X_n(\mathbb{C})$ correspond to certain isomorphism classes of complexes

$$E^\bullet = [E^0 \longrightarrow E^1] \in \mathbf{Cpx}(\mathbb{P})$$

with $E^0 \cong \mathcal{O}_{\mathbb{P}}^{2n+2(k+1)}$ and $E^1 \cong \mathcal{O}_{\mathbb{P}}(1)^{k+1}$. A complex of vector bundles

$$\mathcal{E}^\bullet = [\mathcal{E}^0 \longrightarrow \mathcal{E}^1] \in \mathbf{Cpx}(\mathbb{P} \times X_n)$$

is a *Poincaré family* if for every closed point $x \in X_n(\mathbb{C})$, the corresponding isomorphism class contains the restriction of \mathcal{E}^\bullet to $\mathbb{P} \times \{x\}$.

There is a universal family b^{univ} of isomorphisms b , parameterized by $\text{GL}(S^n V \otimes W)$; cf. Remark 3.5. It induces a complex of vector bundles

$$[E_{n,k}^0 \boxtimes \mathcal{O}_{\text{GL}(S^n V \otimes W)} \xrightarrow{p_{b^{\text{univ}}}} E_{n,k}^1 \boxtimes \mathcal{O}_{\text{GL}(S^n V \otimes W)}]$$

over $\mathbb{P} \times \text{GL}(S^n V \otimes W)$. On this complex, $\text{GL}(V) \times \text{GL}(W)$ acts; the image of \mathbb{G}_m under ι_n acts with weight $-k$.

If k is even, we can tensor the complex with the 1-dimensional representation $\det^{k/2}(V)$ of $\text{GL}(V) \times \text{GL}(W)$. After that, the image of \mathbb{G}_m under ι_n acts trivially, so the action of $\text{GL}(V) \times \text{GL}(W)$ descends to an action of G_n . Then the complex descends to a complex \mathcal{E}^\bullet over $\mathbb{P} \times X_n$, which is indeed a Poincaré family in the above sense.

If k is odd, then one can show that no Poincaré family \mathcal{E}^\bullet over $\mathbb{P} \times X_n$ exists; cf. [22, Proposition 8.5].

One way to view this is to note that the moduli stack parameterizing the complexes in question does depend on k . It is in fact the stack quotient $\mathcal{X}_{n,k}$ of $\text{GL}(S^n V \otimes W)$ modulo the group $\text{GL}(V) \times \text{GL}(W)/\iota_n(\mu_k)$, which can be proved along the same lines as Proposition 3.3.

The stack $\mathcal{X}_{n,k}$ is birational to an affine space times the stack quotient of $\text{End}(V)^2$ modulo $\text{GL}(V)/\mu_k$; cf. the previous remark. Thus the obstruction against Poincaré families is k times the Brauer class over $\mathbb{C}(X_n)$ coming from the generic quaternion algebra. So we see again that the obstruction vanishes if and only if k is even.

Theorem 4.9. *For $n \geq 1$ and $k \geq 1$, the coarse moduli space $M_{\mathbb{P}}(k+1)$ of special $(k+1)$ -instanton bundles over $\mathbb{P} = \mathbb{P}^{2n+1}$ is rational.*

Proof. Recall that $M_{\mathbb{P}}(k+1)$ is constructed as an open subvariety in the Grassmannian bundle $\text{Gr}_k(C^{(2n+k)})$ over X_n . We have just seen that

X_n is rational. Using [10, Proposition 2.2], it follows that $\mathrm{Gr}_k(C^{(2n+k)})$ is also rational, since k and $2n + k$ have the same parity. \square

Remark 4.10. In the special case $n = 1$, which means $\mathbb{P} = \mathbb{P}^3$, the rationality of $M_{\mathbb{P}}(k + 1)$ has already been proved by Hirschowitz and Narasimhan [10, Théorème 4.10]. They also prove the rationality of X_1 [10, Théorème 3.4.II]; their proof is different from the one given here.

Remark 4.11. What [10, Proposition 2.2] proves is that the Grassmannian bundle $\mathrm{Gr}_k(C^{(2n+k)}) \rightarrow X_n$ has rational generic fiber. This does not necessarily mean that it is Zariski-locally trivial. One can show that it is Zariski-locally trivial if and only if k is even; cf. Remark 4.8.

Remark 4.12. [22, Theorem 8.2] states that there is a Poincaré family of special instanton bundles \mathcal{E} over $\mathbb{P} \times M_{\mathbb{P}}(k + 1)$ if and only if k is even. In fact the obstruction class is the pullback of the obstruction class on X_n explained in Remark 4.8.

An equivalent way to state this is to say that the moduli stack $\mathcal{M}_{\mathbb{P}}(k + 1)$ of special instanton bundles is birational to an affine space times the stack $\mathcal{X}_{n,k}$ in Remark 4.8. Observing that $\mathcal{M}_{\mathbb{P}}(k + 1)$ is open in a Grassmannian bundle over $\mathcal{X}_{n,k}$, this can also be proved using Lemma 5.5 and Lemma 4.10 in [11].

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FREIE UNIVERSITÄT BERLIN, INSTITUT FÜR MATHEMATIK, ARNIMALLEE 3,
14195 BERLIN, GERMANY

E-mail address: `norbert.hoffmann@fu-berlin.de`